MATH 312
Lecture 18

EIGENSTUFF

Last Time:

Determinants
- NAM conditions
- equal zero for non-invertible matrices
- $\det(AB) = \det(A)\det(B)$
- Effect of row & column operations
- $\det(A^T) = \det(A)$
- Cofactor expansions & Inverses.

Cofactor Recap: If A is $n \times n$, let A_{ij} be the $(n-1)\times(n-1)$ matrix one gets by removing row i and column j from A, eg.

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \quad A_{12} = \begin{bmatrix} d & f \\ g & i \end{bmatrix}, \quad A_{11} = \begin{bmatrix} e & f \\ h & i \end{bmatrix}$$

$$A_{32} = \begin{bmatrix} a & c \\ d & f \end{bmatrix} \text{ etc.}$$

Set $d_{ij} = (-1)^{i+j}\det(A_{ij})$, and define the "cofactor matrix" C whose (i,j) entry is d_{ij}. Then,

$$\boxed{AC^T = \det(A)\cdot Id} \quad \text{for } \underline{\underline{\text{all}}}\ A,$$

and if A is invertible then

$$\boxed{A^{-1} = C^T/\det(A)} \longleftarrow \text{Formula for inverse!}$$

Eg: $A = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 5 & -2 \\ -1 & 3 \end{bmatrix}$, $C^T = \begin{bmatrix} 5 & -1 \\ -2 & 3 \end{bmatrix}$

and $A^{-1} = \frac{1}{13}\begin{bmatrix} 5 & -1 \\ -2 & 3 \end{bmatrix}$

eg 2
(Harder)

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 4 \end{bmatrix}$$

$$\left.\begin{array}{lll} d_{11} = \begin{vmatrix} -1 & 0 \\ 0 & 4 \end{vmatrix} & d_{12} = -\begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix} & d_{13} = \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} \\ d_{21} = -\begin{vmatrix} 0 & 1 \\ 0 & 4 \end{vmatrix} & d_{22} = \begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix} & d_{23} = -\begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} \\ d_{31} = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} & d_{32} = -\begin{vmatrix} 3 & 1 \\ 2 & 0 \end{vmatrix} & d_{33} = \begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} \end{array}\right\} \leftarrow \begin{array}{c} 3\times 3 = 9 \\ \text{Cofactors} \\ [\text{Hardest part}] \end{array}$$

So, cofactor matrix $C = \begin{bmatrix} -4 & -8 & 1 \\ 0 & 11 & 0 \\ 1 & -2 & -3 \end{bmatrix}$, and so

$$C^T = \begin{bmatrix} -4 & 0 & 1 \\ -8 & 11 & -2 \\ 1 & 0 & -3 \end{bmatrix}$$

So, $\det A = \begin{bmatrix} 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ -8 \\ 1 \end{bmatrix} = -12 + 1 = \underline{\underline{-11}}$

or you can try $\det A = \begin{bmatrix} 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 11 \\ 0 \end{bmatrix} = \underline{\underline{-11}}$

Basically: $[i\text{-th row of } A] \begin{bmatrix} i\text{-th} \\ \text{col} \\ \text{of} \\ C^T \end{bmatrix} = \det(A)$

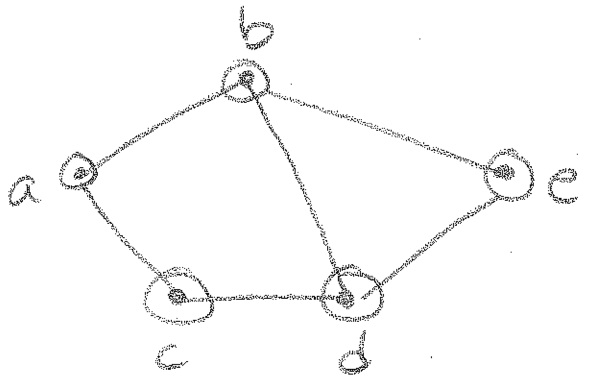
So, $A^{-1} = \frac{1}{\det(A)} C^T$

$$= -\frac{1}{11} \begin{bmatrix} -4 & 0 & 1 \\ -8 & 11 & -2 \\ 1 & 0 & -3 \end{bmatrix}$$

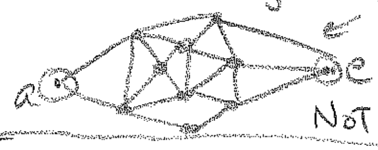
Done! Now, eigenstuff. [Chapter 6]

A "MOTIVATION" FOR EIGENSTUFF!

Here's a graph: How many paths from a to e?



okay, we can count them easily... <u>but</u>



Not here.

Look at adjacency matrix $A = \begin{array}{c} \\ \left[\begin{array}{ccccc} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right] \\ \begin{array}{ccccc} a & b & c & d & e \end{array} \end{array} \begin{array}{c} a \\ b \\ c \\ d \\ e \\ \\ \end{array}$

The entry in row a column d is 1 if and only if you can get from a to d by an edge! So, here we have a zero. BUT:

$$A^2 = \begin{array}{c} \left[\begin{array}{ccccc} 2 & 0 & 0 & 2 & 1 \\ 0 & 3 & 2 & 1 & 1 \\ 0 & 2 & 2 & 0 & 1 \\ 2 & 1 & 0 & 3 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{array}\right] \\ \begin{array}{ccccc} a & b & c & d & e \end{array} \end{array} \begin{array}{c} a \\ b \\ c \\ d \\ e \\ \\ \end{array}$$

There are 2 paths from b to c of length 2! This 2 has come from $(b \xrightarrow{1} d \xrightarrow{1} c) + (b \xrightarrow{1} a \xrightarrow{1} c)$

Similarly, A^3 counts length 3 paths, A^k counts length k paths, etc.

So: might want to compute A^{100}, etc.

— 100^{th} power of matrix
— Count paths in graph

} Both HARD to compute

BUT What if A were DIAGONAL?

$$A = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}, \quad \text{then} \quad A^k = \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix},$$

no trouble at all to compute this!

So | Ask: Is there a NEW basis for $\mathbb{R}^n$ in which the given matrix A is diagonal?

Then, if the answer is "Yes", we have some invertible matrix S so that

$$S^{-1} D S = A,$$

where $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ is DIAGONAL.

Then, $A^k = (S^{-1}DS)^k$

$$= \underbrace{S^{-1}DS \cdot S^{-1}DS \cdot \ldots \cdot S^{-1}DS}_{k \text{ times}}$$

$$= S^{-1} D^k S$$ this is easy!

So, we want to be able to compute A^k efficiently: no counting paths, and no multiplying some complicated matrix with itself dozens of times.

> The ability to EXPONENTIATE, (i.e., raise to powers) square matrices motivates all of the eigen-stuff.

For DIAGONAL matrices, this is easy:

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}, \quad \text{so} \quad D^k = \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix}.$$

So, we want to CHANGE THE BASIS of $\mathbb{R}^n$ so that our matrix A is diagonal in the new basis.

Thus, we seek an INVERTIBLE matrix S so that

$$A = S^{-1} D S, \quad \text{where } D \text{ is diagonal.}$$

If we have this, then A^k becomes easy:

$$A^k = (S^{-1}DS)^k = \underbrace{S^{-1}DS}_{1}\,\underbrace{S^{-1}DS}_{2}\,\underbrace{S^{-1}DS}_{3} \cdots \underbrace{S^{-1}DS}_{k}$$

But SS^{-1} = identity, so

$$\boxed{A^k = S^{-1}D^kS}$$

So: What does a diagonal matrix do to the basis vectors?

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Note $De_1 = \begin{bmatrix} \lambda_1 \\ 0 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda_1 e_1,$

and $De_2 = \begin{bmatrix} 0 \\ \lambda_2 \end{bmatrix} = \lambda_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \lambda_2 e_2.$

So, a diagonal matrix SCALES the i-th basis vector by the i-th diagonal element:

If $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ then

$$De_j = \lambda_j e_j$$

This leads to the EIGENVALUE EQUATION for any square matrix A:

Very important!

Solve $\boxed{A\vec{x} = \lambda\vec{x}}$ for $\lambda, \vec{x}$ unknown.

The λ's which solve this are EIGENVALUES of A. For any eigenvalue $\bar{\lambda}$, the $\vec{x}$'s which solve $A\vec{x} = \bar{\lambda}\vec{x}$ are EIGENVECTORS of A corresponding to $\bar{\lambda}$.